

# EULER SERIES, STIRLING NUMBERS AND THE GROWTH OF THE HOMOLOGY OF THE SPACE OF LONG LINKS.

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**ABSTRACT.** We study the Bousfield-Kan spectral sequence associated to the Munson-Volić cosimplicial model for the space of long links with  $\ell$  strings in  $\mathbb{R}^N$ . We compute explicitly some Euler-Poincaré series associated to the second page of that spectral sequence and deduce exponential growth of their Betti numbers.

## 1. INTRODUCTION

A long link with  $\ell$  strings in  $\mathbb{R}^N$  is a smooth embedding of  $\ell$  disjoint copies of  $\mathbb{R}$  in  $\mathbb{R}^N$  with a fixed behaviour at infinity. This generalizes the classical notion of long knots. In [7] and [8], D. Sinha has constructed a cosimplicial model for the space of long knots, and V. Turchin, I. Volić and the second author have proved in [4] that the associated cohomology Bousfield-Kan spectral sequence collapses at the  $E_2$  page. B. Munson and I. Volić, [6], have constructed a cosimplicial model for the space of long links with  $\ell$  strings, in the same spirit as Sinha's cosimplicial model for the space of long knots. In this paper, we study the cohomology Bousfield-Kan spectral sequence associated to the Munson-Volić cosimplicial model.

Our main result is an explicit closed formula for the Euler characteristic of each line of the second page of that spectral sequence. This computation is based on some interesting relation between the dimensions of the summands of the  $E_2$ -page of the spectral sequence and some Stirling numbers. Since these Euler characteristic have an exponential growth, we deduce an exponential growth of the dimension of the associated graded space of  $E_2$ . We plan to prove in a future paper that the spectral sequence collapses at the  $E_2$  page, generalizing the main result of [4]. This will imply the exponential growth of the Betti numbers of the space of long links, and even of long links modulo  $\ell$ -fold product of the space of long knots (which is a retract of the space of long links, as we will see in this paper.)

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## 2. LONG KNOTS AND LONG STRINGS

Fix  $N \geq 3$ . A long knot is a smooth embedding

$$f: \mathbb{R} \hookrightarrow \mathbb{R}^N$$

satisfying the boundary conditions

$$(2.1) \quad \begin{cases} f([-1, 1]) \subset [-1, 1] \times \mathbb{R}^{N-1} \\ f(t) = (t, 0, \dots, 0) \end{cases} \quad \text{if } |t| \geq 1.$$

In other words  $f$  should be thought of as a string parametrized by  $[-1, 1]$  and knotted inside the box  $[-1, 1] \times \mathbb{R}^{N-1}$  with its two extremities attached to the points  $(\pm 1, 0, \dots, 0)$ . The space of all these embeddings, endowed with a suitable topology (the weak  $\mathcal{C}^\infty$ -topology), is denoted by

$$(2.2) \quad \text{Emb}_\partial(\mathbb{R}, \mathbb{R}^N)$$

where the decoration  $\partial$  is a reminder of the boundary conditions (2.1).

We generalize this to multiple strings. Let  $\ell \geq 1$  be an integer and set  $[\ell] := \{0, \dots, \ell - 1\}$  a set of cardinality  $\ell$ . A *long link with  $\ell$  strings* is a smooth embedding

$$g: \mathbb{R} \times [\ell] \hookrightarrow \mathbb{R}^N$$

satisfying the following boundary conditions

$$(2.3) \quad \begin{cases} g([-1, 1] \times [\ell]) \subset [-1, 1] \times \mathbb{R}^{N-1} \\ g(t, i) = (t, i, 0, \dots, 0) \end{cases} \quad \text{if } |t| \geq 1 \text{ and } i = 0, \dots, \ell - 1.$$

We denote the space of all these embeddings by

$$\text{Emb}_\partial(\mathbb{R} \times [\ell], \mathbb{R}^N).$$

For  $\ell = 1$  this space coincide with the space (2.2).

**2.1. The  $\ell$ -fold product of the space of long knots as a retract up to homotopy of the space of long links.** Each of the  $\ell$  strings of a link can be seen as a long knot. More precisely, given a long link  $g \in \text{Emb}_\partial(\mathbb{R} \times [\ell], \mathbb{R}^N)$ , we can take its restriction to the  $i$ th string

$$g_{(i)} := (g|_{\mathbb{R} \times \{i\}}): \mathbb{R} \times \{i\} \hookrightarrow \mathbb{R}^N$$

for  $i = 0, \dots, \ell - 1$ . For  $s \in \mathbb{R}$ , consider the translation of  $\mathbb{R}^N$  in the second coordinate direction

$$(2.4) \quad \tau_s: \mathbb{R}^N \rightarrow \mathbb{R}^N, (x_1, \dots, x_N) \mapsto (x_1, x_2 + s, x_3, x_4, \dots, x_N).$$

Then the composite  $\tau_{-i} \circ g_{(i)}$  is a long knot (after identifying  $\mathbb{R} \times \{i\}$  with  $\mathbb{R}$ .) Hence we get a map

$$(2.5) \quad \begin{aligned} \rho: \text{Emb}_\partial(\mathbb{R} \times [\ell], \mathbb{R}^N) &\longrightarrow \text{Emb}_\partial(\mathbb{R}, \mathbb{R}^N)^{\times \ell} \\ g &\mapsto (\tau_0 \circ g_{(0)}, \dots, \tau_{-(\ell-1)} \circ g_{(\ell-1)}), \end{aligned}$$

which associates to a long link the  $\ell$ -tuple of the restricted long knots.

Actually  $\rho$  is a retraction up to homotopy. Indeed we can define a map in the other direction

$$(2.6) \quad \iota: \text{Emb}_{\partial}(\mathbb{R}, \mathbb{R}^N)^{\times \ell} \rightarrow \text{Emb}_{\partial}(\mathbb{R} \times [\ell], \mathbb{R}^N)$$

which sends an  $\ell$ -tuple of long knots to a link obtained by juxtaposition of these knots, avoiding to link any two strings. Let us define this map  $\iota$  more precisely. Given a long knot  $f$  define its *width* as the number

$$\text{width}(f) := \inf\{r \geq 0 : f([-1, 1]) \subset [-1, 1] \times [-r, r]^{N-1}\},$$

which depends continuously on  $f$ . For  $R > 0$  consider the linear map

$$H_R: \mathbb{R}^N \rightarrow \mathbb{R}^N, (x_1, x_2, \dots, x_N) \mapsto (x_1, R \cdot x_2, \dots, R \cdot x_N)$$

which is the product of the identity map on  $\mathbb{R}$  and a homothety of rate  $R$  on  $\mathbb{R}^{N-1}$ . If  $f$  a long knot then  $H_R \circ f$  is also a long knot and

$$\text{width}(H_R \circ f) = R \cdot \text{width}(f).$$

In particular, for  $R = \min(\text{width}(f), 1/4)$ ,

$$H_{\min(\text{width}(f), 1/4)} \circ f$$

is a long knot of width  $\leq 1/4$ .

We now define the map  $\iota$  of (2.6) as follows. Let  $(f_0, \dots, f_{\ell-1})$  be an  $\ell$ -tuple of long knots. Then we define

$$g = \iota(f_0, \dots, f_{\ell-1})$$

as the long link characterized by

$$g|_{\mathbb{R} \times \{i\}} = \tau_i \circ H_{\min(\text{width}(f_i), 1/4)} \circ f_i,$$

for  $i = 0, \dots, \ell - 1$ , where  $\tau_s$  is the vertical translation (2.4).

The composition

$$(2.7) \quad \text{Emb}_{\partial}(\mathbb{R}, \mathbb{R}^N)^{\times \ell} \xrightarrow{\iota} \text{Emb}_{\partial}(\mathbb{R} \times [\ell], \mathbb{R}^N) \xrightarrow{\rho} \text{Emb}_{\partial}(\mathbb{R}, \mathbb{R}^N)^{\times \ell}$$

is homotopic to the identity map, the homotopy being built using homotopies

$$H_{\lambda + (1-\lambda) \cdot \min(\text{width}(f_i), 1/4)}$$

for  $0 \leq \lambda \leq 1$ . Thus  $\text{Emb}_{\partial}(\mathbb{R}, \mathbb{R}^N)^{\times \ell}$  can be seen as a retract up to homotopy of  $\text{Emb}_{\partial}(\mathbb{R} \times [\ell], \mathbb{R}^N)$ . By abuse of notation we look at  $\iota$  as an inclusion and we consider the *pair of spaces*

$$(\text{Emb}_{\partial}(\mathbb{R} \times [\ell], \mathbb{R}^N), \text{Emb}_{\partial}(\mathbb{R}, \mathbb{R}^N)^{\times \ell}).$$

**2.2. Long embeddings modulo immersions.** As often when dealing with spaces of embeddings, we will consider these spaces *modulo immersions*. More precisely, a *long immersed 1-string* is a smooth immersion

$$f: \mathbb{R} \looparrowright \mathbb{R}^N$$

with boundary condition (2.1) and the space of long immersed 1-strings is denoted by

$$\text{Imm}_{\partial}(\mathbb{R}, \mathbb{R}^N).$$

We have an inclusion

$$\text{Emb}_{\partial}(\mathbb{R}, \mathbb{R}^N) \hookrightarrow \text{Imm}_{\partial}(\mathbb{R}, \mathbb{R}^N)$$

and its homotopy fibre is denoted by

$$\mathcal{K} := \text{hofibre}(\text{Emb}_\partial(\mathbb{R}, \mathbb{R}^N) \hookrightarrow \text{Imm}_\partial(\mathbb{R}, \mathbb{R}^N))$$

and called the *space of long knots modulo immersions*, or, by abuse of terminology in this paper, simply the *space of long knots*. Similarly we can consider long immersed  $\ell$ -strings  $g: \mathbb{R} \times [\ell] \looparrowright \mathbb{R}^N$  and the homotopy fibre

$$\mathcal{L}_\ell := \text{hofibre}(\text{Emb}_\partial(\mathbb{R} \times [\ell], \mathbb{R}^N) \hookrightarrow \text{Imm}_\partial(\mathbb{R} \times [\ell], \mathbb{R}^N))$$

which will be called in this paper the *space of long links with  $\ell$  strings*.

The inclusion  $\iota$  and retraction  $\rho$  can readily be extended to spaces of immersions on which they are homotopy equivalences. Thus we have a commutative diagram

$$\begin{array}{ccccc} \text{Emb}_\partial(\mathbb{R}, \mathbb{R}^N)^{\times \ell} & \xhookrightarrow{\iota} & \text{Emb}_\partial(\mathbb{R} \times [\ell], \mathbb{R}^N) & \xrightarrow{\rho} & \text{Emb}_\partial(\mathbb{R}, \mathbb{R}^N)^{\times \ell} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Imm}_\partial(\mathbb{R}, \mathbb{R}^N)^{\times \ell} & \xhookrightarrow[\simeq]{\iota} & \text{Imm}_\partial(\mathbb{R} \times [\ell], \mathbb{R}^N) & \xrightarrow[\simeq]{\rho} & \text{Imm}_\partial(\mathbb{R}, \mathbb{R}^N)^{\times \ell}. \end{array}$$

Taking the homotopy fibres of the vertical inclusions we get the sequence of spaces

$$(2.8) \quad \mathcal{K}^{\times \ell} \xrightarrow{\iota} \mathcal{L}_\ell \xrightarrow{\rho} \mathcal{K}^{\times \ell}$$

whose composite is homotopic to the identity. We can look at the map  $\iota$  as an inclusion and consider the pair of spaces

$$(\mathcal{L}_\ell, \mathcal{K}^{\times \ell}).$$

Notice also that by the classical Smale argument [9], we have homotopy equivalences

$$\text{Imm}_\partial(\mathbb{R}, \mathbb{R}^N)^{\times \ell} \simeq \text{Imm}_\partial(\mathbb{R} \times [\ell], \mathbb{R}^N) \simeq (\Omega S^{N-1})^{\times \ell}$$

Since the loop space  $\Omega S^{N-1}$  is very well understood, at least rationally, the study of the sequence (2.8) gives most of the information about the pair

$$(\text{Emb}_\partial(\mathbb{R} \times [\ell], \mathbb{R}^N), \text{Emb}_\partial(\mathbb{R}, \mathbb{R}^N)^{\times \ell}).$$

### 3. COSIMPLICIAL MODELS FOR SPACES OF KNOTS AND LINKS

We recall the cosimplicial models for  $\mathcal{K}$  and  $\mathcal{L}_\ell$  given by Sinha [7] and Munson-Volić [6]. These cosimplicial spaces are built out of some compactifications of configuration spaces in  $\mathbb{R}^N$  that we now review.

First define the configuration space

$$\text{Conf}_\partial(p, \mathbb{R}^N) := \{(x_1, \dots, x_p) \in ]-1, 1[ \times \mathbb{R}^{N-1} : x_i \neq x_j \text{ for } i \neq j\}$$

which is a subspace of  $(]-1, 1[ \times \mathbb{R}^{N-1})^p$ . Define, for integers  $1 \leq i < j \leq p$ , the maps

$$\theta_{ij}: \text{Conf}_\partial(p, \mathbb{R}^N) \rightarrow S^{N-1}, \quad x \mapsto \frac{x_j - x_i}{\|x_j - x_i\|}$$

giving the direction between the  $i$ th and  $j$ th components of the configuration  $x$ .

Following [7], define

$$\text{Conf}_\partial\langle p, \mathbb{R}^n \rangle$$

as the closure of the image of the injection

$$\mathrm{Conf}_\partial(p, \mathbb{R}^N) \xrightarrow{(incl, (\theta_{ij}))} ([-1, 1] \times \mathbb{R}^{N-1})^p \times \prod_{1 \leq i < j \leq p} S^{N-1}.$$

Notice that if we project the subspace

$$\mathrm{Conf}_\partial(p, \mathbb{R}^n) \subset ([-1, 1] \times \mathbb{R}^{N-1})^p \times (S^{N-1})^{\binom{p}{2}}$$

to the factor  $(S^{N-1})^{\binom{p}{2}}$  we get exactly the  $p$ th term of the Kontsevich operad defined by Sinha in [8].

These spaces form a cosimplicial space  $\mathrm{Conf}_\partial(\bullet, \mathbb{R}^n)$  with cofaces given as maps

$$d_i: \mathrm{Conf}_\partial(p, \mathbb{R}^n) \rightarrow \mathrm{Conf}_\partial(p+1, \mathbb{R}^n)$$

corresponding to “doubling” some points of the configuration, when  $1 \leq i \leq p$ ; the first and last cofaces,  $d_0$  and  $d_{p+1}$ , are defined by inserting (or doubling) a point at  $(\mp 1, 0, \dots, 0) \in [-1, 1] \times \mathbb{R}^{N-1}$ . This cosimplicial space models the space of long knots (see Theorem 3.1 below.)

Similarly Munson-Volić in [6] define a cosimplicial space  $\mathrm{Conf}_\partial(\ell\bullet, \mathbb{R}^n)$  which models the space of links with  $\ell$  strings. Its  $p$ th term is

$$\mathrm{Conf}_\partial(\ell p, \mathbb{R}^n)$$

consisting of “virtual” configurations  $(x_{k,r})_{1 \leq k \leq p, 0 \leq r \leq \ell-1}$ . The cofaces, for  $1 \leq i \leq p$

$$d_i: \mathrm{Conf}_\partial(\ell p, \mathbb{R}^n) \rightarrow \mathrm{Conf}_\partial(\ell(p+1), \mathbb{R}^n)$$

consisting in doubling the  $\ell$  components  $x_{i,r}$  (for  $r = 0, \dots, \ell-1$ ) of the configuration.

The maps  $\iota$  and  $\rho$  from (2.6) and (2.5) can readily be generalized to configuration spaces and we get maps of cosimplicial spaces

$$(\mathrm{Conf}_\partial(\bullet, \mathbb{R}^n))^{\times \ell} \xrightarrow{\iota} (\mathrm{Conf}_\partial(\ell\bullet, \mathbb{R}^n)) \xrightarrow{\rho} (\mathrm{Conf}_\partial(\bullet, \mathbb{R}^n))^{\times \ell}$$

whose composite is homotopic to the identity map.

The *homotopical totalization* of a cosimplicial space  $X^\bullet$ ,

$$\mathrm{hoTot}(X^\bullet)_m$$

is the mapping space

$$\mathrm{map}_\Delta(\widetilde{\Delta}^\bullet, X^\bullet)$$

where  $\widetilde{\Delta}^\bullet$  is a variation of the standard cosimplicial space  $\Delta^\bullet$  (more precisely  $\widetilde{\Delta}^\bullet$  is a cofibrant replacement of the standard cosimplicial space  $\Delta^\bullet$  in the projective Quillen model structure, see [5, Section 15].) Equivalently as the totalization of a functorial fibrant replacement  $\widehat{X}^\bullet$  of  $X^\bullet$ ,

$$\mathrm{hoTot}(X^\bullet) = \mathrm{Tot}(\widehat{X}^\bullet) = \mathrm{map}_\Delta(\Delta^\bullet, \widehat{X}^\bullet).$$

We define similarly the homotopy totalization of a map of cosimplicial spaces.

**Theorem 3.1** (Sinha, Munson-Volić). *Assume that  $N \geq 4$ . Applying  $\mathrm{hoTot}$  to the sequence*

$$(\mathrm{Conf}_\partial(\bullet, \mathbb{R}^n))^{\times \ell} \xrightarrow{\iota} (\mathrm{Conf}_\partial(\ell\bullet, \mathbb{R}^n)) \xrightarrow{\rho} (\mathrm{Conf}_\partial(\bullet, \mathbb{R}^n))^{\times \ell}$$

*gives a sequence of spaces weakly equivalent to*

$$\mathcal{K}^{\times \ell} \xrightarrow{\iota} \mathcal{L}_\ell \xrightarrow{\rho} \mathcal{K}^{\times \ell}.$$

*Proof.* The fact that the homotopy totalizations of  $(\text{Conf}_\partial(\bullet, \mathbb{R}^n))^{\times \ell}$  and  $\text{Conf}_\partial(\ell \cdot \bullet, \mathbb{R}^n)$  are  $\mathcal{K}^{\times \ell}$  and  $\mathcal{L}_\ell$  is the content of Sinha and Munson-Volić papers. Sinha's proof for the model of long knots can be summarized as follows. First restrict the cosimplicial to its  $p$ th skeleton and note that the homotopy totalisation of that skeleton is equivalent to the homotopy limit of a punctured  $p$ -cubical diagram of configuration spaces. This cubical diagram is connected by a zigzag of weak equivalences to a cubical diagram whose spaces are essentially partial embedding spaces

$$\text{Emb}_\partial([-1, 1] \setminus \cup_j I_{i_j}, [-1, 1] \times \mathbb{R}^{N-1})$$

where the  $I_i$  are  $p$  disjoint intervals inside  $] - 1, 1[$ . Goodwillie-Klein multiple disjunction theory implies then that this homotopy limit is equivalent (up to some degree going to infinity with  $p$ ) to the space of embeddings

$$\text{Emb}_\partial([-1, 1], [-1, 1] \times \mathbb{R}^{N-1}) = \text{Emb}_\partial(\mathbb{R}, \mathbb{R}^N).$$

The proofs for long links is completely analogous.

It is easy to check that at each level of the above zigzag equivalences we can generalize the maps  $\iota$  and  $\rho$ . This proves the result  $\square$

#### 4. THE COHOMOLOGY SPECTRAL SEQUENCE OF A COSIMPLICIAL SPACE AND POINCARÉ AND EULER SERIES

Let  $X^\bullet$  be a cosimplicial space. There is an associated first-quadrant spectral sequence converging (under some hypothesis as in Proposition 4.1 below) to the cohomology of its homotopical totalization

$$\{E_r^{p,q}\}_{r \geq 1} \implies H^{q-p}(\text{hoTot}(X^\bullet))$$

with coefficients in a fixed field  $\mathbb{K}$ . The  $E_1$  page of this spectral sequence is given by

$$(4.1) \quad E_1^{p,q} := \frac{H^q(X^p)}{\sum_{i=1}^p \text{im}(H^*(s_i))}$$

where

$$s_i: X^p \rightarrow X^{p-1}$$

are the codegeneracies of the cosimplicial space. This is the *cohomology Bousfield-Kan spectral sequence*, or  $H^*$  BKSS for short, see [1]. Sometimes, when  $T$  is the homotopy totalization of the cosimplicial space  $X^\bullet$ , we will denote this spectral sequence by

$$E_r^{p,q}\{T\}$$

to emphasize the underlying cosimplicial space (assuming that the  $T$  makes clear the cosimplicial space we are talking about.)

We recall now a convergence condition for the  $H^*$  BKSS. We say that a positive real number  $\alpha > 0$  is a *lower slope* for a first quadrant spectral sequence  $\{E_r^{*,*}\}_{r \geq 1}$  if

$$\forall p, q \in \mathbb{N} : \quad q < \alpha p \implies E_1^{p,q} = 0.$$

This means that the  $E_1$  page is concentrated above a line of that slope.

**Proposition 4.1** (Bousfield[1]). *If the  $H^*$  BKSS has a lower slope  $\alpha > 1$  then it converges to  $H^*(\text{hoTot}(X^\bullet))$ .*

We also say that a real number  $\beta > 0$  is an *upper slope* for a first quadrant spectral sequence  $\{E_r^{*,*}\}_{r \geq 1}$  if

$$\forall p, q \in \mathbb{N} : \quad q > \beta p \implies E_1^{p,q} = 0.$$

The following result is folklore and easy to prove from the definition (4.1).

**Proposition 4.2.**

$$\dim E_1^{p,q} = \sum_{r=0}^p (-1)^{r-p} \binom{p}{r} \dim H^q(X^r),$$

where  $\binom{p}{r}$  is the binomial coefficient.

We introduce the following notation for Poincaré series. Let  $V^* = \bigoplus_{k=0}^{\infty} V^k$  be a graded vector space. Then its Poincaré series is denoted by

$$(4.2) \quad \underline{V^*}[x] := \sum_{k=0}^{\infty} \dim(V^k) x^k.$$

Here the underlined asterisque  $\underline{*}$  is a reminder that the Poincaré series is taken along this gradation, and the argument in brackets  $[x]$  is the variable of the Poincaré series. For example, the usual Poincaré series of a space  $X$  is

$$H^*(X)[x] = \sum_{q=0}^{\infty} \dim H^q(X) x^q.$$

For a bigraded vector space, as  $E_1^{*,*}$  we will consider the double Poincaré series

$$E_1^{\underline{*}, \underline{*}}[u, x] := \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \dim(E_1^{p,q}) u^p x^q,$$

the simply underlined asterisk is the degree ( $p$ ) corresponding to the *first* variable ( $u$ ), and the doubly underlined asterisque is the degree ( $q$ ) corresponding to the *second* variable ( $x$ ).

Using notation (4.2), Proposition 4.2 implies the following

**Corollary 4.3.** *For the  $H^*$  BKSS of a cosimplicial space  $X^\bullet$ ,*

$$E_1^{p,*}[x] = \sum_{i=0}^p (-1)^{i-p} \binom{p}{i} H^*(X^i)[x].$$

We introduce now a notation for the Euler characteristics and the “Euler series”. For a graded vector space of finite total dimension,  $V^* = \bigoplus_{k=0}^{\infty} V^k$ , its Euler characteristic is denoted by

$$\chi(V^\bullet) := \sum_{k=0}^{\infty} (-1)^k \dim(V^k),$$

hence the bullet is the degree along which is taken the Euler characteristic. Thus for a bigraded object  $E^{*,*}$  we set

$$\chi(E^{\bullet,q}) := \sum_{p=0}^{\infty} (-1)^p \dim(E^{p,q}).$$

We define the *Euler series* of a bigraded object (of finite type) as

$$\chi(E^{\bullet,\underline{*}})[x] := \sum_{q=0}^{\infty} \left( \sum_{p=0}^{\infty} (-1)^p \dim(E^{p,q}) \right) x^q.$$

In other words we have

$$\chi(E^{\bullet,*})[x] = E^{\bullet,*}[-1, x].$$

From Proposition 4.2 we deduce

**Proposition 4.4.** *Assume that the  $H^*$  BKSS of  $X^\bullet$  has a lower slope  $\alpha > 1$  and an upper slope  $\beta < \infty$  and each space  $X^i$  is of finite type. Then*

$$\chi(E_1^{\bullet,*})[x] = \sum_{p,i \geq 0} (-1)^i \binom{p}{i} (H^*(X^i)[x]).$$

*Proof.* First we show that the double series on the right is well defined. This is a consequence of the fact that  $\binom{p}{i} = 0$  for  $i > p$ , the bounds on the slopes, and the finite type hypothesis. The equality is then a direct consequence of Corollary 4.3.  $\square$

One advantage of the Euler series of  $E_1$  is that it can give lower bounds on the Betti numbers of the homotopy totalization when the spectral sequence collapses at the  $E_2$ -term, as we explain now.

Assume that the  $H^*$  BKSS has a lower slope  $\alpha > 1$  and is of finite type, that is  $\dim E_1^{p,q} < \infty$  for all  $p, q$ . The totalization of the  $r$ th page of the spectral sequence is the graded vector space  $(\text{Tot } E_r^{\bullet,*})$  defined by

$$\text{Tot}(E_r^{\bullet,*})^n = \bigoplus_{p \geq 0} E_r^{p,p+n} = \bigoplus_{p=0}^{n/(\alpha-1)} E_r^{p,p+n},$$

which is finite dimensional. When  $r = \infty$  or when the spectral sequence collapses at page  $E_r$ , for some  $r \in \mathbb{N}$ , then, by Proposition 4.1,

$$H^n(\text{hoTot } X^\bullet) \cong \text{Tot}(E_r^{\bullet,*})^n.$$

Since the differential on the page  $E_1$  is horizontal, that is of bidegree  $(-1, 0)$ , the Euler characteristics of the lines of the spectral sequence on pages  $E_1$  and  $E_2$  are the same:

$$\chi(E_1^{\bullet,q}) = \chi(E_2^{\bullet,q}).$$

We then have

**Proposition 4.5.** *Let  $E_1^{\bullet,*}$  be a spectral sequence of finite type and with a lower slope  $\alpha > 1$ . Then*

$$\sum_{k=\lceil n(1-1/\alpha) \rceil}^n \dim(\text{Tot}(E_2^{\bullet,*})^k) \geq |\chi(E_1^{\bullet,n})|.$$

where we denote by  $\lceil x \rceil$  the smallest integer  $\geq x$ .

The previous lemma implies that lower bounds for the Betti numbers of the homotopy totalization of a cosimplicial space  $X^\bullet$  can be obtained, when the  $H^*$  BKSS collapses at the  $E_2$ -page, by computing Euler characteristics at the level of the  $E_1$ -page.

## 5. EULER POINCARÉ SERIES IN THE $H^*$ BKSS FOR SPACES OF LINKS.

Here is our central result.



**Theorem 5.1.** *Let  $N \geq 4$  and consider the  $H^*$  BKSS of the Munson-Volić cosimplicial model for the space of long links with  $\ell$  strings.. Then its first page  $E_1^{*,*}\{\mathcal{L}_\ell\}$  has a lower slope  $> 1$ , a finite upper slope and the Euler series*

$$\chi(E_1^{*,*}\{\mathcal{L}_\ell\})[x] = \frac{1}{(1-x^{N-1})(1-2x^{N-1})\dots(1-\ell x^{N-1})}.$$

In order to prove this theorem, consider first the Poincaré series of configuration spaces in  $\mathbb{R}^N$ . Fadell and Neuwirth in [2] have computed that

$$(5.1) \quad H^*(\text{Conf}(k, \mathbb{R}^N))[x] = (1-x^{N-1})(1-2x^{N-1})\dots(1-(k-1)x^{N-1}).$$

This Poincaré series is closely related to Stirling numbers that we quickly review now. They are integer numbers

$$\begin{bmatrix} n \\ k \end{bmatrix} \quad \text{and} \quad \begin{Bmatrix} n \\ k \end{Bmatrix},$$

named *Stirling numbers of the first and of the second kind*, and defined for example in [3, Section 6.1]. The convention about the signs of these numbers varies in the literature, and we choose the convention of [3] where all these numbers are non-negative. These numbers are defined not only for non-negative integers arguments,  $k, n \in \mathbb{N}$ , but are extended to all integers by

$$\begin{bmatrix} k \\ n \end{bmatrix} = \begin{Bmatrix} k \\ n \end{Bmatrix} = 0 \text{ when } k \text{ and } n \text{ have opposite signs}$$

and

$$\begin{bmatrix} k \\ n \end{bmatrix} = \begin{Bmatrix} -n \\ -k \end{Bmatrix},$$

(see [3, (6.33)])

We have then

$$(5.2) \quad \sum_{j=0}^k \begin{bmatrix} k \\ k-j \end{bmatrix} u^j = (1+u)(1+2u)\dots(1+(k-1)u),$$

and replacing  $u$  by  $x^{N-1}$  in the right hand side we get the Poincaré series of  $\text{Conf}(k, \mathbb{R}^N)$ , see (5.1).

We also have the following generating series for the Stirling numbers of the second kind,

$$(5.3) \quad \sum_{j=0}^{\infty} \begin{Bmatrix} k+j \\ k \end{Bmatrix} u^j = \frac{1}{(1-u)(1-2u)\dots(1-ku)}.$$

We establish the following relation between Stirling numbers of the first and second kind, which will be the key for our main result. This combinatorial result is maybe new. The below proof is due to Manuel Kauers.

**Proposition 5.2.** *For  $j, \ell \geq 1$ ,*

$$\sum_{p=0}^{\infty} \sum_{r=0}^p (-1)^r \binom{p}{r} \begin{bmatrix} \ell r \\ \ell r - j \end{bmatrix} = \begin{Bmatrix} \ell + j \\ \ell \end{Bmatrix}.$$

The left hand side make sense because almost all terms in the exterior infinite sum are 0, as we will see in the proof.

In order to prove this proposition, we need the following lemma.

**Lemma 5.3** (M. Kauers). *Let  $q$  be a polynomial. Then*

$$\sum_{p=0}^{\infty} \left( \sum_{r=0}^p (-1)^r \binom{p}{r} q(r) \right) = q(-1)$$

where all but finitely many terms of the exterior infinite sum in the left hand side are 0.

*Proof of Lemma 5.3.* Consider the generating series

$$(5.4) \quad s(x) := \sum_{p=0}^{\infty} \left( \sum_{r=0}^p (-1)^r \binom{p}{r} q(r) \right) x^p.$$

We need to prove that  $s(x)$  is a finite sum and that  $s(1) = q(-1)$ . We first will prove it for the special polynomials  $q = q_{(d)}$ , for  $d \in \mathbb{N}$ , with

$$(5.5) \quad q_{(d)}(r) := (r+1)(r+2)\cdots(r+d),$$

that is  $q_{(0)}(r) = 1$ ,  $q_{(1)}(r) = r+1$ ,  $q_{(2)}(r) = r^2 + 3r + 2$ , ... Then we will deduce the general case by linearity.

Since  $q_{(0)}(r) = 1$ , we have

$$\sum_{r=0}^{\infty} q_{(0)}(r) y^r = \frac{1}{1-y}.$$

Taking the  $d$ th derivative on both sides we get

$$(5.6) \quad \sum_{r=0}^{\infty} q_{(d)}(r) y^r = \frac{d!}{(1-y)^{d+1}}.$$

The series

$$a(y) := \sum_{r=0}^{\infty} q(r) y^r$$

is the Euler transform (or binomial transform) of the generating series  $s(x)$  of (5.4). Therefore

$$(5.7) \quad s(x) = \frac{1}{1-x} a\left(\frac{x}{x-1}\right).$$

When  $q = q_{(d)}$ , we deduce from (5.6) and (5.7) that

$$\begin{aligned} s(x) &= \frac{1}{1-x} \cdot \frac{d!}{\left(1 - \frac{x}{x-1}\right)^{d+1}} \\ &= d!(1-x)^d. \end{aligned}$$

Thus, for  $q = q_{(d)}$ ,  $s(x)$  is a polynomial, hence a finite sum. Moreover

$$s(1) = \begin{cases} 1 & \text{if } d = 0 \\ 0 & \text{if } d \geq 1, \end{cases}$$

and we compute from (5.5) that

$$q(-1) = \begin{cases} 1 & \text{if } d = 0 \\ 0 & \text{if } d \geq 1. \end{cases}$$

Therefore  $s(1) = q(-1)$  when  $q = q_{(d)}$ , which implies the lemma for those polynomials.

Any polynomial  $q$  is a linear combination of the polynomials  $q_{(0)}, q_{(1)}, q_{(2)}, \dots$ , and both sides of the equation in the statement of the lemma are linear in the polynomials  $q$ . Therefore the lemma is true for all polynomials  $q$ .  $\square$

*Proof of Proposition 5.2 (M. Kauers).* Fix  $j, \ell \geq 1$ . Notice that

$$q(r) := \begin{bmatrix} \ell r \\ \ell r - j \end{bmatrix}$$

is a polynomial in  $\ell r$ , hence in  $r$ , because of the following equation from [3, (6.44) in Section 6.2]:

$$\begin{bmatrix} x \\ x - n \end{bmatrix} = \sum_{i=0}^n \langle\langle n \rangle\rangle_i \binom{x+i}{2n}$$

where  $\langle\langle n \rangle\rangle_i$  are the Euler numbers of the second order, and

$$\binom{x+i}{2n} = \frac{(x+i)(x+i-1)\cdots(x+i-2n+1)}{(2n)!}$$

is a polynomial in  $x$  of degree  $2n$ .

By Lemma 5.3, we deduce that the left hand side of the equation in the statement of the proposition is

$$q(-1) = \begin{bmatrix} -\ell \\ -\ell - j \end{bmatrix},$$

which is the same as  $\{\ell+j\}_\ell$  by [3, (6.33) in Section 6.1].  $\square$

*Proof of Theorem 5.1.* Consider the Munson-Volić cosimplicial space  $\text{Conf}_\partial(\ell\bullet, \mathbb{R}^N)$ . By (5.1) and (5.2), the Poincaré series of the  $p$ th term of the cosimplicial space is

$$\begin{aligned} \text{H}^*(\text{Conf}_\partial(\ell p, \mathbb{R}^N))[x] &= \prod_{j=1}^{\ell p-1} (1 + j x^{N-1}) \\ &= \sum_{j=0}^{\ell p} \begin{bmatrix} \ell p \\ \ell p - j \end{bmatrix} (x^{N-1})^j \\ (5.8) \qquad &= \sum_{j \geq 0} \begin{bmatrix} \ell p \\ \ell p - j \end{bmatrix} (x^{N-1})^j. \end{aligned}$$

Using Proposition 4.4, (5.8), Proposition 5.2, and (5.3), we compute that the Euler series of the first page of the  $H^*$  BKSS is

$$\begin{aligned}
\chi(E_1^{\bullet,*})[x] &= \sum_{p=0}^{\infty} \sum_{r=0}^p (-1)^r \binom{p}{r} H^*(\text{Conf}_{\partial}(\ell r, \mathbb{R}^N))[x] \\
&= \sum_{p=0}^{\infty} \sum_{r=0}^p (-1)^r \binom{p}{r} \sum_{j \geq 0} \left[ \begin{matrix} \ell r \\ \ell r - j \end{matrix} \right] (x^{N-1})^j \\
&= \sum_{j \geq 0} \left( \sum_{p=0}^{\infty} \sum_{r=0}^p (-1)^r \binom{p}{r} \left[ \begin{matrix} \ell r \\ \ell r - j \end{matrix} \right] \right) (x^{N-1})^j \\
&= \sum_{j \geq 0} \left\{ \begin{matrix} \ell + j \\ j \end{matrix} \right\} (x^{N-1})^j \\
&= \frac{1}{(1 - x^{N-1})(1 - 2x^{N-1}) \dots (1 - \ell x^{N-1})}.
\end{aligned}$$

□

**Corollary 5.4.** *Let  $N \geq 4$  and consider the  $H^*$  BKSS of the cosimplicial  $\ell$ -fold product of Sinha's model for the space of long knots. Then its first page  $E_1^{*,*}\{\mathcal{K}^\ell\}$  has a lower slope  $> 1$ , a finite upper slope and the Euler series*

$$\chi(E_1^{\bullet,*}\{\mathcal{K}^\ell\})[x] = \frac{1}{(1 - x^{N-1})^\ell}$$

*Proof.* Given a cosimplicial space  $X^\bullet$ , one can consider its  $\ell$ -fold product  $(X^\bullet)^{\times \ell}$  which is the cosimplicial space defined with its  $p$ th term given by

$$(X^p)^{\times \ell} = X^p \times \dots \times X^p.$$

and the obvious cofaces and codegeneracies induced by those on  $X^\bullet$ .

By the Kunnetth formula, the Euler series of the  $E_1$ -page of the  $H^*$  BKSS of  $(X^\bullet)^{\times \ell}$  is then the  $\ell$ -th power of the Euler series for  $X^\bullet$ . Since the cosimplicial model for long knots is exactly the cosimplicial models for long links with 1 string and since the Euler series for those have been computed in Theorem 5.1 to be

$$\frac{1}{(1 - x^{N-1})},$$

the corollary follows. □

We finish this paper by some application to the growth of the dimension of the totalisation of the  $E_2$  page of a spectral sequence computing

$$H^*(\mathcal{L}_\ell, \mathcal{K}^{\times \ell}).$$

By (2.8), we have a pair of spaces  $(\mathcal{L}_\ell, \mathcal{K}^{\times \ell})$  in which the subspace is a retract (up to homotopy) of the bigger space  $\mathcal{L}_\ell$ . Since the retraction is at the level of the cosimplicial models, we have a spectral sequence  $E_r^{*,*}\{(\mathcal{L}_\ell, \mathcal{K}^{\times \ell})\}$  which converges to  $H^*(\mathcal{L}_\ell, \mathcal{K}^{\times \ell})$ , and the spectral sequence is a quotient

$$E_r^{*,*}\{(\mathcal{L}_\ell, \mathcal{K}^{\times \ell})\} \cong E_r^{*,*}\{\mathcal{L}_\ell\} / E_r^{*,*}\{\mathcal{K}^{\times \ell}\}.$$

Therefore we have

$$\chi(E_2^{\bullet,*}\{(\mathcal{L}_\ell, \mathcal{K}^{\times\ell})\})[x] = \chi(E_2^{\bullet,*}\{\mathcal{L}_\ell\})[x] - \chi(E_2^{\bullet,*}\{\mathcal{K}^{\times\ell}\})[x].$$

By Theorem 5.1 and Corollary 5.4 we deduce that

$$\chi(E_2^{\bullet,*}\{(\mathcal{L}_\ell, \mathcal{K}^{\times\ell})\})[x] = \frac{1}{(1-x^{N-1})(1-2x^{N-1})\dots(1-\ell x^{N-1})} - \frac{1}{(1-x^{N-1})^\ell}.$$

The coefficients of that series have an exponential growth of rate  $(\ell)^{1/(N-1)} > 1$ . By Proposition 4.5 the Betti numbers of the totalization of the  $E_2$  page of the spectral sequence have the same growth.

We conjecture that this spectral sequence collapses at the  $E_2$ -page (this is known to be the case for the spectral sequence for long knots, and probably the same method will work for the space of long links, hence also for the spectral sequence of the pair  $(\mathcal{L}_\ell, \mathcal{K}^{\times\ell})$ .)

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